Integral transforms for discounting value and cash flows

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Variables:

R_t and r(t) :	the cash flow R at time t measured in currency units [cu] and the flow rate r measured in currency units per time [cu/y]
t:	time, discrete or continuous
N:	periods of the investment project, or date of the end of the investment project.
i:	yearly interest rate
j:	imaginary unit: $j^2 = -1$

Net present value (NPV):

discrete:

$$\mathrm{NPV}(i) = \sum_{t=0}^{\infty} \frac{R_t}{(1+i)^t}$$

continuous:

$$\mathrm{NPV}(i) = \int_{t=0}^{\infty} (1+i)^{-t} \cdot r(t)\, dt$$

with R_t and r(t) = 0 if t>N (no flows when the period under review is over)

$$(1+i)^{-t} = e^{\ln(1+i)^{-t}} = e^{-t\cdot\ln(1+i)} = e^{-t\cdot s}$$ with s = ln(1+i)

$$(1+i)^{-t} = e^{-t\cdot s} = z^{-t}$$ with z = 1+i = e^s

The NPV formula can be written as:

$$\mathrm{NPV}(z) = \sum_{t=0}^{\infty} R_t \cdot z^{-t}$$

$$\mathrm{NPV}(s) = \int_{t=0}^{\infty} e^{-st} \cdot r(t)\, dt$$

Usually, economical calculus chooses values for interest rates i or derivatives z and s from the real number space. The two-dimensional number space of the complex plane is known to mathematicians for centuries and has been proven to be useful in the analysis of system dynamics. Thus, it is suggested to expand the discount rate in the NPV calculation with an imaginary interest component. Laplace and Z-transforms are well known tools in system analysis.

Let s and z be of the set of complex numbers:
$z = x + yj = A\cdot e^{j\varphi}$ and $s = a + bj$
$A=|z|= \sqrt{(x^2+y^2)}$, $\varphi = \arg(z)$, $-\pi < \varphi \leq \pi$

$$z^{-t} = (x+yj)^{-t} = (A\cdot e^{j\varphi})^{-t} \cdot = A^{-t}\cdot e^{-j\varphi t} \qquad e^{-st} = e^{-(a+bj)t} = e^{-at-bjt} = e^{-at}\cdot e^{-jbt}$$

The real component is responsible for interpreting steady growth or declining trends in the given time series, whereas the imaginary component gives answers on cyclic characteristics, resulting from the rotating vector $e^{-j\varphi t}$. Back transformation of z and s in the domain of interest rates is done as follows.

i = z-1 = (x-1) + yj

$i = e^s - 1 = (e^a - 1) + e^{bj}$